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QUASI-RESIDUATED MAPPINGS AND BAER ASSEMBLIES.(U)

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(6) QUASI-RESIDUATED MAPPINGS AND BAER ASSEMBLIES

by

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Quasi-residuated Mappings and Baer Assemblies. By T.S. Blyth, Mathematical Institute, University of St.Andrews *and* W.C. Hardy, Centre for Naval Analyses, Arlington, Virginia.

ABSTRACT

SYNOPSIS

We consider, for a given ordered set E with minimum element 0, the semigroup Q of 0-preserving isotone mappings on E and examine necessary and sufficient conditions under which an element $f \in Q$ is such that the left [resp. right] annihilator of f in Q is a principal left [resp. right] ideal of Q generated by a particular type of idempotent. The results obtained lead us to introduce the concept of a Baer assembly which we use to extend to the case of a semilattice the Baer semigroup coördinatisation theory of lattices. We also derive a coördinatisation of particular types of semilattice.

in which point f belongs to set Q

ABSTRACT

1. Introduction.

In the publications [1] and [2] a profound connection is established between the theory of lattices and the theory of semigroups. We begin by describing a slight generalisation of this, a detailed account of which will be found in [3]. Let S be a semigroup and let K be a two-sided ideal of S. For each $x \in S$ define the *left* and *right K-annihilator of* x by

$$L_K(x) = \{y \in S;\ yx \in K\}, \qquad R_K(x) = \{y \in S;\ xy \in K\}.$$

We say that the pair $<S;K>$ forms a *Baer semigroup* if and only if, for each $x \in S$, $L_K(x)$ is a principal idempotent-generated left ideal of S and $R_K(x)$ is a principal idempotent-generated right ideal of S. If $<S;K>$ is a Baer semigroup then the sets $\mathcal{L}(S) = \{L_K(x);\ x \in S\}$ and $\mathcal{R}(S) = \{R_K(x);\ x \in S\}$ form dually isomorphic

lattices. To proceed in the opposite direction, let E and F be bounded ordered sets; *i.e.*, ordered sets which have a minimum element and a maximum element. A mapping $f : E \to F$ is said to be *residuated* if and only if it is isotone (order-preserving) and such that there exists a (necessarily unique) mapping $f^+ : F \to E$ which is isotone and such that $f^+ \circ f \geq id_E$ and $f \circ f^+ \leq id_F$. When such a mapping f^+ exists, it is given by the prescription

$$f^+(y) = \max\{x \in E;\ f(x) \leq y\}.$$

It can be shown that $f : E \to F$ is residuated if and only if the pre-image under f of each principal order ideal of F (*i.e.*, a subset of the form $[\leftarrow, x] = \{y \in F;\ y \leq x\}$) is a principal order ideal of E. The set $Res(E)$ of residuated mappings from E to itself forms a semigroup under composition of mappings and an investigation into just when $\langle Res(E);\{0\}\rangle$ forms a Baer semigroup yields the fundamental result : *if E is a bounded ordered set then the following are equivalent :*

(1) E is a lattice;

(2) $\langle Res(E);\{0\}\rangle$ is a Baer semigroup;

(3) there is a Baer semigroup $\langle S;K\rangle$ such that $E \simeq \mathcal{R}(S)$.

When situation (3) holds we say that $\langle S;K\rangle$ *coördinatises* E.

In this paper we shall extend this result to the case of a semilattice. This we shall accomplish by examining annihilators in a much larger semigroup, namely the semigroup of isotone mappings which admit 0 as a fixed point. We obtain our coördinatisation theorem using the concept of a Baer assembly and we apply

this to obtain coördinatisations of particular semilattices.

2. Quasi-residuated mappings.

Let E and F be ordered sets. We shall say that a mapping $f : E \to F$ is *quasi-residuated* if and only if it is isotone and such that there exists a mapping $h : F \to E$ such that $f \circ h \leq id_E$. It can be shown that f is quasi-residuated if and only if it is isotone and such that $(\forall\, y \in F)\ \{x \in E;\ f(x) \leq y\} \neq \emptyset$. In the case where E has a minimum element 0 it is easily seen that $f : E \to E$ is quasi-residuated if and only if it is isotone and such that $f(0) = 0$.

In what follows we shall assume that E has a minimum element 0 and we shall denote by Q the set of quasi-residuated mappings on E. It is clear that Q forms a semigroup under composition of mappings and that this semigroup admits a zero element, namely the zero mapping on E which we shall also denote by 0 without confusion. For each $f \in Q$ we shall write

$$L(f) = \{g \in Q;\ g \circ f = 0\}, \qquad R(f) = \{g \in Q;\ f \circ g = 0\}.$$

Clearly $L(f)$ is a left ideal of Q and $R(f)$ is a right ideal of Q. Let us now examine under what conditions these annihilators are idempotent-generated principal ideals.

Definitions. We shall say that $f \in Q$ *has a principal kernel* if and only if $\{x \in E;\ f(x) = 0\}$ admits a maximum element which we shall denote by $f^+(0)$; and that $f \in Q$ *has a bounded image* if and only if $\{f(x);\ x \in E\}$ admits a maximum element which we shall denote

by $f(\pi)$. [Note that if E also has a maximum element π then by the isotone property *every* $f \in Q$ has a bounded image.]

THEOREM 1 *Given $f \in Q$, the following conditions are equivalent:*

(1) f has a principal kernel;

(2) $R(f)$ is a principal right ideal generated by an idempotent which has a bounded image.

Proof. (1)$\Rightarrow$(2) : Suppose first that f has a principal kernel. For each $y \in E$ define a mapping $\theta_y : E \to E$ by

$$\theta_y(x) = \begin{cases} x & \text{if } x \leq y; \\ y & \text{if } x \not\leq y. \end{cases}$$

Clearly each θ_y is isotone and idempotent. Moreover, each $\theta_y \in Q$ and has a bounded image. Now by the definition of $f^+(0)$ we have

$$\left(f \circ \theta_{f^+(0)}\right)(x) = \begin{cases} f(x) = 0 & \text{if } x \leq f^+(0); \\ f[f^+(0)] = 0 & \text{if } x \not\leq f^+(0), \end{cases}$$

so that $f \circ \theta_{f^+(0)}$ is the zero map. Hence $\theta_{f^+(0)} \in R(f)$ and consequently $\theta_{f^+(0)} \circ Q \subseteq R(f)$. On the other hand, if $g \in Q$ then

$$\begin{aligned} g \in R(f) &\Rightarrow (\forall x \in E) \quad f[g(x)] = 0 \\ &\Rightarrow (\forall x \in E) \quad g(x) \leq f^+(0) \\ &\Rightarrow g = \theta_{f^+(0)} \circ g \\ &\Rightarrow g \in \theta_{f^+(0)} \circ Q \end{aligned}$$

and so we deduce that $R(f) = \theta_{f^+(0)} \circ Q$ which gives (2).

(2)$\Rightarrow$(1) : If now (2) holds, say $R(f) = \zeta \circ Q$ where $\zeta = \zeta \circ \zeta$, let $\zeta(\pi)$ denote the greatest element of $\operatorname{Im} \zeta$. Since, for $x \in E$,

$$(f \circ \theta_x)(y) = f[\theta_x(y)] = \begin{cases} f(y) & \text{if } y \leq x; \\ f(x) & \text{if } y \not\leq x, \end{cases}$$

we see, using the fact that f is isotone, that

$$\theta_x \in R(f) \iff f(x) = 0.$$

We thus have

$$\begin{aligned} f(x) = 0 &\Rightarrow \theta_x \in R(f) = \zeta \circ Q \\ &\Rightarrow \theta_x = \zeta \circ \theta_x \\ &\Rightarrow x = \theta_x(x) = \zeta[\theta_x(x)] \leq \zeta(\pi). \end{aligned}$$

Since clearly $\zeta \in R(f)$ we have $f[\zeta(\pi)] = 0$. It thus follows that f has a principal kernel with $f^+(0) = \zeta(\pi)$.

THEOREM 2 *Given $f \in Q$, the following conditions are equivalent:*

(1) f has a bounded image;

(2) $L(f)$ is a principal left ideal generated by an idempotent which has a principal kernel.

Proof. (1)$\Rightarrow$(2) : Suppose that f has a bounded image. For each $y \in E$ define a mapping $\psi_y : E \to E$ by

$$\psi_y(x) = \begin{cases} 0 & \text{if } x \leq y; \\ x & \text{if } x \nleq y. \end{cases}$$

Clearly each ψ_y is isotone and idempotent. Moreover, each $\psi_y \in Q$ and has a principal kernel. Since by definition $f(\pi)$ is the greatest element of $\operatorname{Im} f$, we see easily that $\psi_{f(\pi)} \circ f$ is the zero map and so $\psi_{f(\pi)} \in L(f)$ whence $Q \circ \psi_{f(\pi)} \subseteq L(f)$. On the other hand, if $g \in Q$ then

$$\begin{aligned} g \in L(f) &\Rightarrow (\forall x \in E) \quad g[f(x)] = 0 \\ &\Rightarrow g[f(\pi)] = 0 \\ &\Rightarrow \{x \leq f(\pi) \Rightarrow g(x) = 0 = g(0)\} \\ &\Rightarrow g = g \circ \psi_{f(\pi)} \in Q \circ \psi_{f(\pi)} \end{aligned}$$

and so we deduce that $L(f) = Q \circ \psi_{f(\pi)}$ which gives (2).

(2)$\Rightarrow$(1) : Suppose that (2) holds, say $L(f) = Q \circ \xi$ where $\xi = \xi \circ \xi$ and $\xi^+(0)$ exists. Since $\xi \circ f$ is the zero map, we have

$$(*) \qquad (\forall x \in E) \qquad f(x) \leq \xi^+(0).$$

The property (1) will follow from this if we can show that $\xi^+(0)$ is an element of $\operatorname{Im} f$. This we do as follows :

(a) suppose that f is the zero map : then clearly $Q = L(f) = Q \circ \xi$ from which it follows that there exists $h \in Q$ such that $id_E = h \circ \xi$. Hence we have $\xi(x) = 0 \Rightarrow x = 0$ and consequently $\xi^+(0) = 0 = f(0) \in \operatorname{Im} f$.

(b) suppose that f is not the zero map : then (*) shows that $\xi^+(0) \neq 0$ so we can define a mapping $t : E \to E$ by setting

$$(**) \qquad t(x) = \begin{cases} 0 & \text{if } (\exists z \in \operatorname{Im} f) \ x \leq z \ ; \\ \xi^+(0) & \text{otherwise.} \end{cases}$$

To see that t is isotone, let $x \leq y$; then

(i) if $(\exists z \in \operatorname{Im} f)$ $x \leq z$ and $(\exists z^* \in \operatorname{Im} f)$ $y \leq z^*$ then we have $t(x) = 0 = t(y)$;

(ii) if $(\exists z \in \operatorname{Im} f)$ $x \leq z$ and $(\forall z^* \in \operatorname{Im} f)$ $y \nleq z^*$ then we have $t(x) = 0 < \xi^+(0) = t(y)$;

(iii) the case where $(\forall z \in \operatorname{Im} f)$ $x \nleq z$ and $(\exists z^* \in \operatorname{Im} f)$ $y \leq z^*$ cannot arise since by hypothesis $x \leq y$;

(iv) if $(\forall z \in \operatorname{Im} f)$ $x \nleq z$ and $(\forall z^* \in \operatorname{Im} f)$ $y \nleq z^*$ then we have $t(x) = \xi^+(0) = t(y)$.

Thus in all possible cases $t(x) \leq t(y)$ and so t is isotone. Moreover, $t(0) = 0$ and so $t \in Q$. Noting that if $x \in \operatorname{Im} f$ then $t(x) = 0$, we see that $t \in L(f) = Q \circ \xi$ and so $t = t \circ \xi$. Thus

$$t[\xi^+(0)] = (t\circ\xi)[\xi^+(0)] = t(0) = 0.$$

It follows from (**) that there exists $x^* \in E$ with $\xi^+(0) \leq f(x^*)$ and from (*) that $f(x^*) \leq \xi^+(0)$. Hence $\xi^+(0) = f(x^*) \in \mathrm{Im}\, f$.

THEOREM 3 *Let $f \in Q$ have a principal kernel. Then for each idempotent $\zeta \in Q$ such that $R(f) = \zeta \circ Q$ the greatest element of $\mathrm{Im}\,\zeta$ exists and is $f^+(0)$.*

Proof. Consider the mapping $\theta_{f^+(0)}$ as defined in Theorem 1. Clearly $\theta_{f^+(0)} \in R(f)$ and so, if $R(f) = \zeta \circ Q$ with $\zeta = \zeta \circ \zeta$, we have $\theta_{f^+(0)} = \zeta \circ \theta_{f^+(0)}$. Consequently

$$f^+(0) = \theta_{f^+(0)}[f^+(0)] = \left(\zeta\circ\theta_{f^+(0)}\right)[f^+(0)] = \zeta[f^+(0)].$$

Now $f \circ \zeta = 0$ and so $(\forall\, x \in E)$ $\zeta(x) \leq f^+(0)$. It follows from this that $f^+(0)$ is the maximum element of $\mathrm{Im}\,\zeta$.

THEOREM 4 *Let $f \in Q$ have a bounded image. Then for each idempotent $\xi \in Q$ such that $L(f) = Q \circ \xi$ the element $\xi^+(0)$ exists and is $f(\pi)$.*

Proof. Consider the mapping $\psi_{f(\pi)}$ as defined in Theorem 2. Clearly $\psi_{f(\pi)} \in L(f)$ and so, if $L(f) = Q \circ \xi$ where $\xi = \xi \circ \xi$, we have $\psi_{f(\pi)} = \psi_{f(\pi)} \circ \xi$. Consequently

$$\xi(x) = 0 \Rightarrow \psi_{f(\pi)}(x) = 0 \Rightarrow x \leq f(\pi).$$

But clearly if $x \leq f(\pi)$ then $\xi(x) \leq \xi[f(\pi)] = 0$. It therefore follows that $\xi^+(0)$ exists and is none other than $f(\pi)$.

The results of Theorems 3 and 4 give rise to the following two results which we shall be able to express in a useful equivalent way in the next section.

THEOREM 5 *If E is an ordered set with minimum element 0 define*

$$\mathcal{R} = \{R(f);\ f \in Q \text{ and } f \text{ has a principal kernel}\}$$

and order $\mathcal{R}$ by set inclusion. Then the ordered sets $E, \mathcal{R}$ are isomorphic.

Proof. By virtue of Theorem 3 we can define a mapping $\delta : \mathcal{R} \to E$ by the prescription $\delta[R(f)] = f^+(0)$ which can be written in the equivalent form $\delta[\zeta \circ Q] = \zeta(\pi)$ where ζ is an idempotent such that $R(f) = \zeta \circ Q$. Now $\zeta \circ Q \subseteq \eta \circ Q$ implies that $\zeta = \eta \circ \zeta$ which gives $\operatorname{Im} \zeta \subseteq \operatorname{Im} \eta$ and hence $\zeta(\pi) \leq \eta(\pi)$. On the other hand, if $\zeta(\pi) \leq \eta(\pi)$ and if $\eta \circ Q = R(g)$ where $g \in Q$ has a principal kernel then by Theorem 3 we have $\eta(\pi) = g^+(0)$ and consequently $\zeta(\pi) \leq g^+(0)$. It follows that $g \circ \zeta = 0$ so that $\zeta \in R(g) = \eta \circ Q$ and hence $\zeta \circ Q \subseteq \eta \circ Q$. We thus have

$$\delta(\zeta \circ Q) \leq \delta(\eta \circ Q) \iff \zeta(\pi) \leq \eta(\pi) \iff \zeta \circ Q \subseteq \eta \circ Q.$$

To show that δ is an order isomorphism, it therefore suffices to show that it is surjective. For this purpose, we note that for each $x \in E$ the mapping ψ_x as defined in Theorem 2 has a principal kernel and that $\psi_x^+(0) = x$. Thus $\delta[R(\psi_x)] = x$ and so δ is surjective.

THEOREM 6 *If E is an ordered set with minimum element 0 define*

$$\mathcal{L} = \{L(f);\ f \in Q \text{ and } f \text{ has a bounded image}\}$$

and order $\mathcal{L}$ by set inclusion. Then the ordered sets $E, \mathcal{L}$ are dually isomorphic.

Proof. By virtue of Theorem 4 we can define a mapping $\gamma : \mathcal{L} \to E$

by the prescription $\gamma[L(f)] = f(\pi)$ which can be written in the equivalent form $\gamma(Q\circ\xi) = \xi^+(0)$ where ξ is an idempotent such that $L(f) = Q\circ\xi$. Now $Q\circ\xi \subseteq Q\circ\eta$ implies that $\xi = \xi \circ \eta$ from which we deduce that $\eta(x) = 0 \Rightarrow \xi(x) = 0$ and so $\eta^+(0) \leq \xi^+(0)$. On the other hand, if $\eta^+(0) \leq \xi^+(0)$ and if $Q\circ\eta = L(g)$ where $g \in Q$ has a bounded image then by Theorem 4 we have $\eta^+(0) = g(\pi)$ and hence $\xi \circ g = 0$. Thus $\xi \in L(g) = Q \circ \eta$ and so $Q \circ \xi \subseteq Q \circ \eta$. We thus have

$$\gamma(Q\circ\xi) \leq \gamma(Q\circ\eta) \iff \xi^+(0) \leq \eta^+(0) \iff Q\circ\xi \supseteq Q\circ\eta.$$

To show that γ is a dual isomorphism it therefore suffices to show that it is surjective. For this purpose, we note that for each $x \in E$ the mapping θ_x as defined in Theorem 1 has a bounded image and that $\theta_x(\pi) = x$. Thus $\gamma[L(\theta_x)] = x$ and so γ is surjective.

3. Baer assemblies.

Definition. Let A be a semigroup, let B and C be distinguished subsets of A and let K be a two-sided ideal of A. We shall say that $<A;B,C;K>$ forms a *Baer assembly* if and only if the following properties are satisfied :

$$\begin{cases} (\forall x \in B)(\exists e = e^2 \in C) & L_K(x) = Ae\ ; \\ (\forall y \in C)(\exists f = f^2 \in B) & R_K(y) = fA\ . \end{cases}$$

By way of example, we mention that any Baer semigroup $<S;K>$ may be regarded as a Baer assembly $<S;S,S;K>$. Also, if E is an ordered set with a minimum element 0, if Q denotes the semigroup of quasi-residuated mappings on E and if B,C are the subsets of Q con-

sisting respectively of those mappings which have bounded images, principal kernels then Theorems 1 and 2 show that $\langle Q;B,B;\{0\}\rangle$ is a Baer assembly.

We shall say that a Baer assembly $\langle A;B,C;K\rangle$ is *normal* if and only if $K \subseteq B \cap C$. Any Baer semigroup is clearly a normal Baer assembly. Likewise, if E is an ordered set with a minimum element and a maximum element and if Q,B,C are defined as above then we have $0 \in B \cap C$ and $B = Q$ so that $\langle Q;Q,C;\{0\}\rangle$ is a normal Baer assembly.

THEOREM 7 *If $\langle A;B,C;K\rangle$ is a normal Baer assembly then*

(1) A has an identity element which belongs to $B \cap C$;

(2) K is a principal ideal generated by a central idempotent.

Proof. (1) By hypothesis each $k \in K$ satisfies $k \in B \cap C$ and so there exist idempotents $e \in C$, $f \in B$ such that $A = L_K(k) = Ae$ and $A = R_K(k) = fA$. These equalities show that e is a right identity for A and that f is a left identity for A. It follows that $e = f$ and is a two-sided identity for A; we denote this as usual by 1. Since $e \in C$ and $f \in B$ we have $1 \in B \cap C$.

(2) Since $1 \in B \cap C$ there exist idempotents g,h such that $K = L_K(1) = Ag$ and $K = R_K(1) = hA$. These equalities show that g is a right identity for K and that h is a left identity for K. Hence $g = h$ is an identity for K which we shall denote by $k°$. We thus have $K = Ak° = k°A$. Now for each $x \in A$ we have $xk° \in K$ and so $xk° = k°xk°$. Likewise $k°x \in K$ and so $k°x = k°xk°$. It is immediate that $k°$ is an idempotent which belongs to the centre of A.

In the next result we list further fundamental properties of normal Baer assemblies, by way of preparation for which we make the following remarks. If A is an arbitrary semigroup then it is well known that the power set $P(A)$, equipped with the multiplication defined by

$$\begin{cases} XY = \{xy;\ x\in X,\ y\in Y\}; \\ X\emptyset = \emptyset = \emptyset X, \end{cases}$$

is a residuated semigroup in which the residuals are given by

$$X^{\cdot}.Y = \{z\in A;\ (\forall\ y\in Y)\ zy\in X\}, \qquad X.^{\cdot}Y = \{z\in A;\ (\forall\ y\in Y)\ yz\in X\}.$$

Suppose now that K is a non-empty subset of A and consider the mappings $\hat{L}_K, \hat{R}_K : P(A) \to P(A)$ given by the prescriptions

$$\hat{L}_K(X) = K^{\cdot}.X, \qquad \hat{R}_K(X) = K.^{\cdot}X.$$

We have :

THEOREM 8 *If A is a semigroup and K is a non-empty subset of A then*

(1) $\hat{L}_K \circ \hat{R}_K \circ \hat{L}_K = \hat{L}_K$ *and* $\hat{R}_K \circ \hat{L}_K \circ \hat{R}_K = \hat{R}_K$.

If moreover $\langle A;B,C;K\rangle$ *is a normal Baer assembly and if we define* $\mathcal{L}(B) = \{L_K(x);\ x\in B\}$ *and* $\mathcal{R}(C) = \{R_K(x);\ x\in C\}$ *then*

(2) if $e = e^2\in B$ *then* $eA\in\mathcal{R}(C) \iff eA = (\hat{R}_K\circ\hat{L}_K)(eA)$ *and if* $f = f^2\in C$ *then* $Af\in\mathcal{L}(B) \iff Af = (\hat{L}_K\circ\hat{R}_K)(Af)$;

(3) the restriction $\vec{L}_K$ *of* $\hat{L}_K$ *to* $\mathcal{R}(C)$ *is a dual order isomorphism of* $\mathcal{R}(C)$ *onto* $\mathcal{L}(B)$ *whose inverse is* $\vec{R}_K$, *the restriction of* $\hat{R}_K$ *to* $\mathcal{L}(B)$.

Proof. (1) Since

$$\hat{R}_K(X) = K.^{\cdot}X \supseteq Y \iff XY\subseteq K \iff X\subseteq K^{\cdot}.Y = \hat{L}_K(Y),$$

we see that, when considered as a mapping from $P(A)$ to its dual, $\hat{R}_K$ is residuated with residual given by $(\hat{R}_K)^+ = \hat{L}_K$. The equalities in (1) are then an immediate consequence of $\hat{L}_K \circ \hat{R}_K \geq id_{P(A)}$ and $\hat{R}_K \circ \hat{L}_K \geq id_{P(A)}$ [which express the fact that the pair $(\hat{R}_K, \hat{L}_K)$ sets up a Galois connection on $P(A)$].

(2) Given $e = e^2 \in B$, suppose first that $eA \in \mathcal{R}(C)$; say $eA = R_K(x)$ where $x \in C$. Noting that $R_K(x) = \hat{R}_K\{x\}$ we have, by (1),

$$(\hat{R}_K \circ \hat{L}_K)(eA) = (\hat{R}_K \circ \hat{L}_K \circ \hat{R}_K)\{x\} = \hat{R}_K\{x\} = R_K(x) = eA.$$

Conversely, suppose that $e = e^2 \in B$ with $eA = (\hat{R}_K \circ \hat{L}_K)(eA)$. If we let $L_K(e) = Ae^*$ where e^* is an idempotent in C then, since A has an identity element by Theorem 7, $\hat{L}_K(eA) = L_K(e)$ and so

$$eA = (\hat{R}_K \circ \hat{L}_K)(eA) = \hat{R}_K[L_K(e)] = \hat{R}_K(Ae^*) = R_K(e^*) \in \mathcal{R}(C).$$

(3) It follows from the above that $\vec{R}_K \circ \vec{L}_K = id_{\mathcal{R}(C)}$ and that $\vec{L}_K \circ \vec{R}_K = id_{\mathcal{L}(B)}$. Since $\vec{L}_K$ and $\vec{R}_K$ are each antitone, the result follows.

Remark. In the proof of (2) above we used the fact that, since A has an identity element, $\hat{L}_K(eA) = L_k(e)$. In what follows we shall make use of this fact several times without reference.

Definition. If E is an ordered set then we shall say that E is *coördinatised* by a normal Baer assembly $\langle A;B,C;K\rangle$ if and only if $E \simeq \mathcal{R}(C)$ or, equivalently [by Theorem 8(3)], $E \overset{d}{\simeq} \mathcal{L}(B)$ where $\overset{d}{\simeq}$ denotes dual order isomorphism.

With this terminology we see that the results of §2 can be succinctly summarised as follows : *if E is an ordered set with minimum element 0, if Q denotes the semigroup of quasi-residuated*

mappings on E *and if* B, C *denote respectively the subsets of* Q *consisting of those mappings which have bounded images, principal kernels then* $\langle Q; B, C; \{0\}\rangle$ *is a normal Baer assembly which coördinatises* E.

Having observed this fundamental result, we can now explicitly state the ultimate goal of the theory we are developing : to isolate those Baer assemblies which coördinatise ordered sets of a given type. In the following sections we begin the pursuit of this goal by restricting our attention to semilattices. During the course of this, the Baer semigroup coördinatisation of lattices will emerge in a rather nice way.

4. Coördinatisation of $\wedge$-semilattices.

THEOREM 9 *An ordered set* E *is an* $\wedge$*-semilattice with maximum element* π *if and only if, for each* $x \in E$, $\theta_x \in Res(E)$. *In this case the residual map* θ_x^+ *is given by*

$$\theta_x^+(y) = \begin{cases} \pi & \text{if } y \geq x; \\ y \wedge x & \text{if } y \not\geq x. \end{cases}$$

Proof. By definition, we have

$$\theta_x(y) = \begin{cases} y & \text{if } y \leq x; \\ x & \text{if } y \not\leq x. \end{cases}$$

Now if $x \leq z$ or $x \geq z$ then clearly $x \wedge z$ exists. Suppose then that $x \not\leq z$ and $x \not\geq z$. Given $y \in E$, we have

$$\theta_x(y) \leq z \Rightarrow \theta_x(y) \neq x \Rightarrow \theta_x(y) = y \Rightarrow \begin{cases} y \leq x; \\ y \leq z, \end{cases}$$

and conversely if $y \leq x,z$ then $\theta_x(y) = y \leq z$. Thus

$$\{y\ ;\ \theta_x(y) \leq z\} = [\leftarrow,x] \wedge [\leftarrow,z],$$

and so $x \wedge z$ exists if and only if $\max\{y;\ \theta_x(y) \leq z\}$ exists. If we observe that $\mathrm{Im}\ \theta_x = [\leftarrow,x]$, we see also that $\max\{y;\ \theta_x(y) \leq x\}$ exists if and only if π exists. We conclude that E is an $\wedge$-semilattice with maximum element π if and only if each θ_x is residuated. Finally, it is readily verified that the mapping θ_x^+ as given above is isotone and such that $\theta_x^+ \circ \theta_x \geq id_E$ and $\theta_x \circ \theta_x^+ \leq id_E$, so that θ_x^+ is indeed the residual of θ_x.

This result, together with the following, which examines the close relationship between residuated mappings and mappings which have principal kernels, allows us to isolate a Baer assembly which allows a coördinatisation of $\wedge$-semilattices.

THEOREM 10 *Let E be an ordered set with minimum element O and let Q denote the semigroup of quasi-residuated mappings on E. Given $g \in Q$, the following conditions are equivalent :*

(1) if $f \in Q$ has a principal kernel then $f \circ g$ has a principal kernel;

(2) $g \in Res(E)$.

Proof. (1)$\Rightarrow$(2) : By hypothesis $g(0) = 0$ and so, given any $y \in E$, we have $\{x \in E;\ g(x) \leq y\} \neq \emptyset$. Now the mapping ψ_y as defined in Theorem 2 has a principal kernel and so, applying (1), we obtain

$$g(x) \leq y = \psi_y^+(0) \iff (\psi_y \circ g)(x) = 0 \iff x \leq (\psi_y \circ g)^+(0).$$

It follows that g is residuated with g^+ given by the formula

$$g^+(y) = (\psi_y \circ g)^+(0).$$

(2)$\Rightarrow$(1) : If g is residuated and f has a principal kernel then we have

$$(f \circ g)(x) = 0 \iff g(x) \leq f^+(0) \iff x \leq g^+[f^+(0)],$$

from which it follows that $(f \circ g)^+(0)$ exists and is $g^+[f^+(0)]$.

Corollary *If C denotes the subset of Q consisting of those mappings with principal kernels then, in $P(Q)$, $Res(E) = C.^{\cdot}C$.*

Proof. Using the above, we have

$$g \in Res(E) \iff C \circ g \subseteq C \iff g \in C.^{\cdot}C.$$

Definition. By a *right Baer assembly* we shall mean a normal Baer assembly $\langle A;B,D;K\rangle$ in which

(1) B is a subsemigroup of A;

(2) $B \subseteq D$;

(3) in $P(A)$, $B \subseteq C.^{\cdot}C$ [*i.e.*, $CB \subseteq C$].

THEOREM 11 *Let E be an ordered set with minimum element 0. Let Q denote the semigroup of quasi-residuated mappings on E and let C be the subset of Q consisting of those mappings which have principal kernels. Then the following conditions are equivalent :*

(1) E is an $\cap$-semilattice with a maximum element π;

(2) $\langle Q; Res(E),C;\{0\}\rangle$ is a right Baer assembly;

(3) there is a right Baer assembly $\langle A;B,D;K\rangle$ with $E \simeq \mathcal{R}(D)$.

Proof. (1)$\Rightarrow$(2) : Immediate from Theorems 9 and 10.

(2)$\Rightarrow$(3): If (2) holds then (3) follows by taking $A = Q$, $B = Res(E)$, $D = C$, $K = \{0\}$ and applying Theorem 5.

(3)$\Rightarrow$(1): Let $eA, fA \in \mathcal{R}(D)$, say $eA = R_K(s)$ and $fA = R_K(t)$,

e and f being idempotents in B. Since $te \in DB \subseteq D(D.{}^{\cdot}D) \subseteq D$ we have $R_K(te) = gA$ where $g = g^2 \in B$. Thus, from $(te)(eg) = teg \in K$ we deduce that $eg \in R_K(te) = gA$ so that $eg = geg$ and hence eg is an idempotent in B. Now

$$x \in egA \Rightarrow x = egx \Rightarrow \begin{cases} tx = tegx \in K \Rightarrow x \in R_K(t), \\ sx = segx \in seA \subseteq K \Rightarrow x \in R_K(s) \end{cases}$$

$$\Rightarrow x \in R_K(s) \cap R_K(t);$$

and conversely

$$\begin{aligned} x \in R_K(s) \cap R_K(t) \Rightarrow x \in eA &\Rightarrow x = ex \\ &\Rightarrow tex = tx \in K \\ &\Rightarrow x \in R_K(te) = gA \\ &\Rightarrow x = gx \\ &\Rightarrow x = ex = egx \in egA. \end{aligned}$$

We have thus shown that

$$eA \cap fA = R_K(s) \cap R_K(t) = egA.$$

We now show that $egA \in \mathcal{R}(D)$ whence it follows that $\mathcal{R}(D)$ is an $\cap$-semilattice. From $egA \subseteq eA, fA$ we obtain, by Theorem 8(1),(2),

$$\hat{R}_K[L_K(eg)] \subseteq \hat{R}_K[L_K(e)] \cap \hat{R}_K[L_K(f)] = eA \cap fA = egA$$

and, since $eg \in \hat{R}_K[L_K(eg)]$, the reverse inclusion holds. Hence we have $\hat{R}_K[L_K(eg)] = egA$ and so, by Theorem 8(2), $egA \in \mathcal{R}(D)$. Now, by Theorem 7, A has an identity element and for any $k \in K$ we have $R_K(k) = A = 1A$; hence A is the maximum element of $\mathcal{R}(D)$. Since $K = R_K(1)$ is clearly the minimum element of $\mathcal{R}(D)$, we see that $\mathcal{R}(D)$ is a bounded $\cap$-semilattice, whence so also is E.

In the next section we shall develop an analogue of Theorem 11 for the case of a $\cup$-semilattice.

5. Coördinatisation of $\cup$-semilattices.

Let E be a $\cup$-semilattice with minimum element 0, let Q be the semigroup of quasi-residuated mappings on E and let C be the subset of Q consisting of those mappings with principal kernels. Let us now consider the subset T of Q consisting of all the quasi-residuated $\cup$-homomorphisms on E. Clearly T is a subsemigroup of Q which contains the zero map. Since each residuated mapping on a $\cup$-semilattice is necessarily a $\cup$-homomorphism, we have $\mathrm{Res}(E) \subseteq T$. Now each mapping θ_x as defined in Theorem 1 is readily seen to be an element of T and so we deduce from Theorem 1 that, in the semigroup T,

$$(\forall f \in T \cap C) \qquad R(f) = \theta_{f^+(0)} \circ T.$$

Now it is readily verified that in general the mapping ψ_y as defined in Theorem 2 is not an element of $\mathrm{Res}(E)$. But if we consider for each $y \in E$ the mapping $\hat{\psi}_y : E \to E$ defined by

$$\hat{\psi}_y(x) = \begin{cases} 0 & \text{if } x \leq y; \\ x \cup y & \text{if } x \not\leq y, \end{cases}$$

then each $\hat{\psi}_y$ is idempotent. Moreover, as can readily be verified, each $\hat{\psi}_y \in \mathrm{Res}(E)$ with $\hat{\psi}_y^+$ given by

$$\hat{\psi}_y^+(z) = \begin{cases} z & \text{if } z \geq y; \\ y & \text{if } z \not\geq y. \end{cases}$$

Finally, we observe that for each $f \in T$ which has a bounded image we have, in the semigroup T,

$$g \in L(f) \iff (\forall\, x \in E)\quad g[f(x)] = 0$$

$$\iff g[f(\pi)] = 0$$

$$\iff (\forall\, x \in E)\quad \left(g \circ \hat{\psi}_{f(\pi)}\right)(x) = \begin{cases} g(0) = g(x) & \text{if } x \leq f(\pi); \\ g[x \vee f(\pi)] = g(x) & \text{if } x \not\leq f(\pi), \end{cases}$$

$$\iff g \circ \hat{\psi}_{f(\pi)} = g$$

$$\iff g \in T \circ \hat{\psi}_{f(\pi)}.$$

These considerations lead us to the following result in which $\overset{d}{\simeq}$ denotes dual order isomorphism and the term *left Baer assembly* is defined from the previous notion using left/right duality.

THEOREM 12 *Let E be an ordered set with minimum element 0. Then the following conditions are equivalent :*

(1) E is a $\vee$-semilattice with a maximum element π;

(2) there is a left Baer assembly $\langle A;B,D;K\rangle$ with $E \overset{d}{\simeq} \mathcal{L}(B)$.

Proof. (1)$\Rightarrow$(2): The existence of a maximum element implies that the zero map is residuated and that every isotone mapping on E has a bounded image. Since $\mathrm{Res}(E) \subseteq T \wedge C$ the previous considerations show that $\langle T;T,\mathrm{Res}(E);\{0\}\rangle$ is a left Baer assembly. [Note that since T is a semigroup containing id_E we have $T = T^{\bullet}.T$ in $P(T)$.] That $E \overset{d}{\simeq} \mathcal{L}(T)$ follows from the fact that Theorems 2,4,6 carry over *in toto* from Q to the subsemigroup T (with ψ_y replaced by $\hat{\psi}_y$).

(2)$\Rightarrow$(1): This is analogous to the proof of (3)$\Rightarrow$(1) in Theorem 11.

Remark. Observe that when E is a bounded lattice E can be simultaneously coördinatised by the left Baer assembly $\langle T;T,\mathrm{Res}(E);\{0\}\rangle$ and the right Baer assembly $\langle Q;\mathrm{Res}(E),C;\{0\}\rangle$. The latter can be

replaced by $\langle T;\mathrm{Res}(E),\mathrm{Res}(E);\{0\}\rangle$, which is both a right and a left Baer assembly, and an application of Theorems 11 and 12 produce the result : *an ordered set* E *is a bounded lattice if and only if it can be coördinatised by a two-sided Baer assembly.* This result recaptures the Baer semigroup coördinatisation of lattices since the coördinatisation shows that whenever E is a bounded lattice $\mathrm{Res}(E)$ is a Baer semigroup which coördinatises E and if a two-sided Baer assembly is of the form $\langle A;B,B;K\rangle$ then $\langle B;K\rangle$ is necessarily a Baer semigroup.

6. Coördinatisation of Glivenko and Glivenko-Brouwer semilattices.

By a *Glivenko semilattice* (or pseudo-complemented semilattice) we shall mean an $\wedge$-semilattice E with minimum element 0 in which each translation $t_x : y \to x\wedge y$ has a principal kernel. By a *Glivenko-Brouwer semilattice* (implicative semilattice, or relatively pseudo-complemented semilattice) we mean an $\wedge$-semilattice E with minimum element 0 in which each translation is a residuated mapping. By a *Glivenko* [resp. *Glivenko-Brouwer*] *assembly* we shall mean a right Baer assembly $\langle A;B,D;K\rangle$ in which there is an abelian idempotent subsemigroup $\hat{D}$ of D [resp. $\hat{B}$ of B] such that, for each $x\in D$, there exists $e\in\hat{D}$ [resp. $e\in\hat{B}$] with $R_K(x) = eA$ and, for each $e\in\hat{D}$ [resp. $e\in\hat{B}$], $eA\in\mathcal{R}(D)$.

Just as each Glivenko-Brouwer semilattice is in particular a Glivenko semilattice, it follows from the fact that $B\subseteq D$ that each Glivenko-Brouwer assembly is in particular a Glivenko assembly. The

nomenclature we have chosen for these assemblies is justified in the following coördinatisation theorem.

THEOREM 13 *Let* E *be an ordered set with minimum element* 0. *Then the following conditions are equivalent :*

(1) E *is a Glivenko [resp. Glivenko-Brouwer] semilattice;*

(2) E *can be coördinatised by a Glivenko [resp. Glivenko-Brouwer] assembly.*

Proof. (1)$\Rightarrow$(2): If E is a Glivenko [resp. Glivenko-Brouwer] semilattice then each translation t_x has a principal kernel [resp. is residuated] and $\{t_x ; x \in E\}$ is an abelian idempotent semigroup. Q and C being as before, given $f \in C$ and $g \in Q$ we have, since E has a maximum element $\pi = t_0^+(0)$,

$$\begin{aligned} g \in R(f) &\iff f[g(\pi)] = 0 \\ &\iff (\forall x \in E) \quad g(x) \leq f^+(0) \\ &\iff (\forall x \in E) \quad g(x) \wedge f^+(0) = g(x) \\ &\iff g = t_{f^+(0)} \circ g \\ &\iff g \in t_{f^+(0)} \circ Q. \end{aligned}$$

It follows that $R(f) = t_{f^+(0)} \circ Q$ where $t_{f^+(0)} \in C$ [resp. $t_{f^+(0)} \in \mathrm{Res}(E)$], an immediate consequence of which is that $\langle Q;\mathrm{Res}(E),C;\{0\}\rangle$ forms a Glivenko [resp. Glivenko-Brouwer] assembly which coördinatises E.

(2)$\Rightarrow$(1): Suppose that $\langle A;B,D;K\rangle$ is a Glivenko [resp. Glivenko-Brouwer] assembly which coordinatises E. If $e,g \in \hat{D}$ [resp. $e,g \in \hat{B}$] then from $eg = ge$ we deduce that $egA \subseteq eA \cap gA$. But if $h \in \hat{D}$ [resp. $h \in \hat{B}$] is such that $hA \subseteq eA$ and $hA \subseteq gA$ then we have $h = eh =$

$geh = egh$ and so $hA \subseteq egA$. Hence in $\mathcal{R}(D)$ we have $eA \wedge gA = egA$.

In the case of a Glivenko assembly, if $e \in \hat{D} \subseteq D$ then $R_K(e) \in \mathcal{R}(D)$ so that from

$$eA \wedge fA = efA \subseteq K \iff ef \in K \iff f \in R_K(e)$$

we see that the pseudo-residual (pseudo-complement) $(eA)^*$ exists and is none other than $R_K(e)$. Hence E is a Glivenko semilattice.

In the case of a Glivenko-Brouwer assembly, if $e, f, g \in \hat{B}$ and $f^* \in D$ is such that $L_K(f) = Af^*$ then $f^*e \in DB \subseteq D(D.{}^{\cdot}D) \subseteq D$ and so $R_K(f^*e) \in \mathcal{R}(D)$. We then have

$$\begin{aligned} eA \wedge gA = egA \subseteq fA &\Rightarrow eg = egf = feg \\ &\Rightarrow f^*eg = f^*feg \in K \\ &\Rightarrow g \in R_K(f^*e) \\ &\Rightarrow gA \subseteq R_K(f^*e), \end{aligned}$$

and conversely

$$\begin{aligned} gA \subseteq R_K(f^*e) &\Rightarrow f^*eg \in K \\ &\Rightarrow eg \in R_K(f^*) = \hat{R}_K(Af^*) = \hat{R}_K[L_K(f)] = fA \\ &\Rightarrow eA \wedge gA = egA \subseteq fA. \end{aligned}$$

It follows that $t^+_{eA}(fA)$ exists in $\mathcal{R}(D)$ and is $R_K(f^*e)$. Thus $\mathcal{R}(D)$ is a Glivenko-Brouwer semilattice whence so also is E. [Note that in this case we have the same formula for pseudo-residuals as in the Glivenko case; for, taking f to be the central idempotent k° of K [Theorem 7(2)] we can take $f^* = 1$ [Theorem 7(1)] to obtain $t^+_{eA}(k^\circ A) = R_K(e)$.]

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